# Higher Gross–Zagier formula and relative Langlands duality

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# Overview

**Gross–Zagier-type formula:**

$$\text{Intersection numbers} \quad \Longleftrightarrow \quad \text{Derivatives of } L\text{-functions}$$

**Relative Langlands duality:** (Ben-Zvi–Sakellaridis–Venkatesh)

$$G \curvearrowright M \quad \Longleftrightarrow \quad \check{G} \curvearrowright \check{M}$$

$$\text{Period integral attached to } M \quad \Longleftrightarrow \quad L\text{-function attached to } \check{M}$$

Suggested by function-field examples, we propose:
**Arithmetic Relative Langlands duality:**

$$\begin{array}{c}\text{Intersection numbers of special cycles}\\ \text{attached to } M\end{array} \quad \Longleftrightarrow \quad \begin{array}{c}\text{Derivatives of } L\text{-functions attached to}\\ I \in \Gamma(\check{M}, T_{\check{M}})\end{array}$$

We call $I$ the **intersection vector field**.

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# Outline

# Gross–Zagier-type formulas

# Special cycles on Shimura varieties

Consider a map between Shimura data:

$$p : (H, h_H) \to (G, h_G).$$

After choosing compatible level subgroups, this induces a map of Shimura varieties:

$$p_{\mathrm{Sh}} : \mathrm{Sh}_H \to \mathrm{Sh}_G\,.$$

Let $V \in \mathrm{Rep}(\check{G})$ be the highest weight representation associated with $h_G$. The *special cycle* attached to $p$ is

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$$\langle (Z_V)_\pi, (Z_V)_{\pi^\vee} \rangle \sim L'(\pi, \frac{1}{2}),$$

where

- $(Z_V)_\pi \in \mathrm{CH}^*(\mathrm{Sh}_G)_{\pi,0}$ is the $\pi$-isotypic component of $Z_V$.
- $\langle -, - \rangle$ is the Beilinson–Bloch height pairing.
- $L(\pi, s)$ is the relevant $L$-function.

For $G = \mathrm{PGL}_2$ and a non-split maximal torus $H$ (split over an imaginary quadratic field), this recovers the original Gross–Zagier formula.

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# Example 1: GGP cycles

$$H = U(n-2,1), \quad G = U(n-1,1) \times U(n-2,1)$$

$$\check{G} = \mathrm{GL}_n \times \mathrm{GL}_{n-1}, \quad V = \mathrm{Std}_n \boxtimes \mathrm{Std}_{n-1} \in \mathrm{Rep}(\check{G})$$

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**Conjecture (Gan–Gross–Prasad)**

$$\langle (Z_V)_\pi, (Z_V)_{\pi^\vee} \rangle \sim L'(\mathrm{BC}(\pi_n) \times \mathrm{BC}(\pi_{n-1}), \tfrac{1}{2}).$$

Similar formulas: Orthogonal AGGP, AIPF...

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A version of this is proved when $n = 2$ (Chen–Lu–Zhang).

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# Higher Gross–Zagier-type formulas

Special cycles have a parallel function-field theory:

| Number field | Function field |
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| $(G, h_G)$ | $(G, V \in \mathrm{Rep}(\check{G}))$ |
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A *higher Gross–Zagier-type formula* has the form

$$\langle (Z^r_V)_\pi, (Z^r_V)'_{\pi^\vee} \rangle \sim \left(\frac{d}{ds}\right)^r \Big|_{s=0} L(\pi, s).$$

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**Theorem (Liu-W, W25)**

*If $\rho_\pi$ is geometrically irreducible, then*

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*If $\rho_\pi$ is geometrically irreducible, then*

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# Example 1: Rankin–Selberg cycles

$H = \mathrm{GL}_{n-1}, \quad G = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$
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$H =$ split simple reductive group, $\quad G = H \times H$
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$$\langle (Z^r_V)_\pi, (Z^r_V)'_{\pi^\vee} \rangle \sim \left(\frac{d}{ds}\right)^r \Big|_{s=0} L(\pi_H, \mathrm{Ad}, \epsilon_{V_H} s).$$

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$H = G =$ split simple reductive group
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# Relative Langlands duality

# Hyperspherical duality

Let $X = H\backslash G$. Its cotangent bundle $M = T^*X$ is a graded $G$-Hamiltonian space.

If $M$ is *hyperspherical*, Ben-Zvi–Sakellaridis–Venkatesh associate to it a graded $\check{G}$-Hamiltonian space $\check{M}$.

Concretely, $\check{M}$ is a $\check{G} \times \mathbb{G}_{\mathrm{m}}$-variety equipped with:

- a symplectic form $\omega \in \Gamma(\check{M}, \Omega^2_{\check{M}})$ of weight 2;
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# Period integrals and $L$-functions

$\pi$: tempered cuspidal automorphic representation

$\rho_\pi : \mathrm{Gal} \to \check{G}(\mathbb{C})$: $L$-parameter

$\check{M}^{\rho_\pi(\mathrm{Gal})} = \{x\}$ is a single point

$T^*_x\check{M} = \bigoplus_i S_i(d_i) \in \mathrm{Rep}(\mathrm{Gal} \times \mathbb{G}_m)$: cotangent space of $\check{M}$ at $x$

$f \in \pi$: normalized test vector

**Conjecture (Ben-Zvi–Sakellaridis–Venkatesh)**

$$\left| \int_{[H]} f \right|^2 \sim \prod_i L(S_i, d_i/2)$$

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$$\langle (Z_V)_\pi, (Z'_V)_{\pi^\vee} \rangle \sim \frac{d}{ds}\Big|_{s=0} \prod_i L(S_i, \epsilon_{V,i} s + d_i/2).$$

The constants $\epsilon_{V,i}$ are called **eigenweights**, and are determined by an **intersection vector field** $I \in \Gamma(\check{M}, T_{\check{M}})^{\check{G}\times\mathbb{G}_{\mathrm{m}}}$ constructed from $Z_V, Z'_V$.

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# Intersection vector field

# Determining the eigenweights

The special cycles $Z_V, Z'_V$ determine an **intersection vector field**

$$I \in \Gamma(\check{M}, T_{\check{M}})^{\check{G}\times\mathbb{G}_{\mathrm{m}}}.$$

Since $I$ vanishes at $x$, its linearization defines the *Hessian endomorphism*

$$\mathrm{Hess}_{I,x} : T_x^*\check{M} \longrightarrow T_x^*\check{M}.$$

It is $\mathrm{Gal} \times\mathbb{G}_{\mathrm{m}}$-equivariant.
Choose a decomposition $T_x^*\check{M} = \bigoplus_i S_i(d_i)$ such that

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# Construction of the intersection vector field

In our examples,

$$\begin{aligned} Z_V &\Longleftarrow c_V : \check{M} \to V \\ Z'_V &\Longleftarrow c'_V : \check{M} \to V^* \end{aligned}$$

for $\check{G}$-equivariant maps $c_V, c'_V$.

Set

$$f_V := (c_V, c'_V) : \check{M} \to T^*V.$$

Pull back the canonical 1-form $\theta_V \in \Gamma(T^*V, \Omega_{T^*V})$ to define

$$\boxed{I := f_V^*\theta_V \in \Gamma(\check{M}, \Omega_{\check{M}}) \cong \Gamma(\check{M}, T_{\check{M}})}.$$

We require $f_V$ to factor through the isotropic locus:

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# A precise formulation

For an unramified tempered cuspidal automorphic representation $\pi$ over a function field, we expect:

## Conjecture

*Assuming $H^*_c(\mathrm{Sht}^r_{G,V})_\pi \neq 0$ and $S_{\rho_\pi} = Z(\check{G})$, one has*

$$\langle (Z^r_V)_\pi, (Z^r_V)'_{\pi^\vee} \rangle = \frac{q^{(g-1)(-\dim G + \dim H)}}{|Z(\check{G})| L(\mathrm{Ad}_{\rho_\pi}, 1)(\log q)^r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} \left( q^{(2g-2)bs} \prod_i L(S_i, \epsilon_{V,i} s + d_i) \right)$$

*where*

- *$b$ is an explicit constant depending on $c_V, c'_V$;*
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$H = \mathrm{GL}_{n-1}, \quad G = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$

$\check{G} = \mathrm{GL}_n \times \mathrm{GL}_{n-1}, \quad \check{M} = (\mathrm{Std}_n \boxtimes \mathrm{Std}_{n-1} \oplus \mathrm{Std}_n^* \boxtimes \mathrm{Std}_{n-1}^*)(1).$

$V = V^* = \check{M} \in \mathrm{Rep}(\check{G})$

Set

$$c_V = c'_V = \mathrm{id} : \check{M} \to V = V^*$$

For a basis $\check{M} \cong \mathbb{C}^N$, this gives

$$I = \sum_{j=1}^{N} x_j \partial_{x_j} \in \Gamma(\check{M}, T_{\check{M}}).$$

Suppose $\check{M}^{\mathrm{Gal}} = \{0\}$. Then

$$T_0^* \check{M} = V(1) \in \mathrm{Rep}(\mathrm{Gal} \times \mathbb{G}_{\mathrm{m}}).$$

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$$c_V = c'_V = \mathrm{id} : \check{M} \to V = V^*$$

For a basis $\check{M} \cong \mathbb{C}^N$, this gives

$$I = \sum_{j=1}^{N} x_j \partial_{x_j} \in \Gamma(\check{M}, T_{\check{M}}).$$

Suppose $\check{M}^{\mathrm{Gal}} = \{0\}$. Then

$$T_0^* \check{M} = V(1) \in \mathrm{Rep}(\mathrm{Gal} \times \mathbb{G}_{\mathrm{m}}).$$

$$\boxed{\mathrm{Hess}_{I,0} = \mathrm{id} \in \mathrm{End}(V(1))}.$$

# Example 2: Diagonal case

$H =$ split simple algebraic group, $\quad G = H \times H$
$\check{G} = \check{H} \times \check{H}, \quad \check{M} = T^*\check{H} = \check{H} \times \check{\mathfrak{h}}.$
$V = V_H \boxtimes V_H \in \mathrm{Rep}(\check{G})$ for $V_H \in \mathrm{Rep}(\check{H})$ irreducible
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Suppose $\check{M}^{\mathrm{Gal}} = \{x = (1, 0) \in \check{H} \times \check{\mathfrak{h}}\}$. Then

$$T_x^*\check{M} = \check{\mathfrak{h}}^*(2) \oplus \check{\mathfrak{h}}(0) \in \mathrm{Rep}(\mathrm{Gal} \times \mathbb{G}_m).$$

$$\boxed{\mathrm{Hess}_{I,x} = (0, \epsilon_{V_H} \cdot \mathrm{id}) \in \mathrm{End}(\check{\mathfrak{h}}^*(2) \oplus \check{\mathfrak{h}}(0))}.$$

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# Remarks on eigenweights

The eigenweight $\epsilon_{V_H}$ is given by

$$\epsilon_{V_H} = 2h^{\vee} l \frac{\mathrm{tr}(dr_{V_H}(X) dr_{V_H}(Y))}{\kappa(X, Y)}$$

where

- $X, Y \in \check{\mathfrak{h}}$;
- $\kappa : \check{\mathfrak{h}} \times \check{\mathfrak{h}} \to \mathbb{C}$ is the Killing form;
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# Pseudo-example 3: Volume case

$H = G =$ split simple reductive group

$\check{M} = \check{G} \times (e + \check{\mathfrak{g}}_f)$, where $(e, f, h)$ is a principal $\mathfrak{sl}_2$-triple.

Let $V \in \mathrm{Rep}(\check{G})$ be irreducible, with highest weight vectors $v \in V$ and $v^* \in V^*$.

Define

$$c_V : \check{G} \times (e + \check{\mathfrak{g}}_f) \to V, \quad c_V(g, e + X) = gv,$$
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where $d_V = \langle 2\rho, \mu_V \rangle$, $\mu_V \in X_*(T)_+$ is the highest weight of $V$.

Consider $x = (1, e) \in \check{G} \times (e + \check{\mathfrak{g}}_f)$. Then

$$T^*_x \check{M} = \check{\mathfrak{g}}^{*/\!/} \oplus \bigoplus_i \check{\mathfrak{g}}_{f,i}(2d_i) \in \mathrm{Rep}(\mathrm{Gal} \times \mathbb{G}_{\mathrm{m}}).$$

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# Proof strategy in examples

# More on relative Langlands duality

For the spherical $G$-variety $X = H\backslash G$, the unramified Hecke category

$$\mathrm{Shv}(L^+G\backslash LG/L^+G)^{\heartsuit} \simeq \mathrm{Rep}(\check{G})^{\heartsuit},$$

acts on the relative Hecke category $\mathrm{Shv}(LX/L^+G)$.

Consider the basic object

$$\delta_X \in \mathrm{Shv}(LX/L^+G).$$

For $V \in \mathrm{Rep}(\check{G})$, local relative Langlands duality predicts

$$\mathrm{Map}(\check{M}, V)^{\mathbb{G}} \cong \mathrm{Map}(V * \delta_X, \delta_X).$$

Globally, the map $\pi : \mathrm{Bun}_H \to \mathrm{Bun}_G$ yields the *period sheaf*

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# Local–global compatibility

Start with $c_V \in \mathrm{Map}(\check{M}, V)^{/\!\!/} \cong \mathrm{Map}(V * \delta_X, \delta_X)$:

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c_V : V * \delta_X \to \delta_X \arrow[d, dashed, "\text{Geometric } \Theta\text{-series}"] \\
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Verdier duality similarly relates $c_V'$ to $Z_V'$.

# Intersection number as a trace

Let $\mathbb{O}_{\check{M},\rho_\pi}$ be the stalk of the algebra of $L$-observables.
$T_x^*\check{M}^{/\!/} = \bigoplus_i S_i[d_i](d_i/2)$.

$$\mathrm{Gr}_\bullet(\mathbb{O}_{\check{M},\rho_\pi}) \cong \mathsf{Sym}^\bullet R\Gamma(C_{\overline{\mathbb{F}}_q}, T_x^*\check{M}^{/\!/}).$$

The maps $c_V, c_V'$ define an **intersection observable**

$$\mathbb{I}_{\rho_\pi} : \mathbb{O}_{\check{M},\rho_\pi} \to \mathbb{O}_{\check{M},\rho_\pi}$$

such that

$$\langle (Z_V^r)_\pi, (Z_V^r)'_{\pi^\vee} \rangle \sim \mathrm{tr}(\mathrm{Frob} \circ (\mathbb{I}_{\rho_\pi})^r, \mathbb{O}_{\check{M},\rho_\pi}).$$

We expect

$$\mathrm{Gr}_\bullet \mathbb{I}_{\rho_\pi} \sim \nabla_{R\Gamma(\mathrm{Hess}_{l,x})} \in \mathsf{End}(\mathsf{Sym}^\bullet R\Gamma(T_x^*\check{M}^{/\!/})),$$

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Let $\mathbb{O}_{\check{M},\rho_\pi}$ be the stalk of the algebra of $L$-observables.
$T_x^*\check{M}^{/\!/} = \bigoplus_i S_i[d_i](d_i/2)$.

$$\mathrm{Gr}_\bullet(\mathbb{O}_{\check{M},\rho_\pi}) \cong \mathsf{Sym}^\bullet\, R\Gamma(C_{\overline{\mathbb{F}}_q}, T_x^*\check{M}^{/\!/}).$$

The maps $c_V, c'_V$ define an **intersection observable**

$$\mathbb{I}_{\rho_\pi} : \mathbb{O}_{\check{M},\rho_\pi} \to \mathbb{O}_{\check{M},\rho_\pi}$$

such that

$$\langle (Z_V^r)_\pi, (Z_V^r)'_{\pi^\vee}\rangle \sim \mathrm{tr}(\mathrm{Frob}\circ(\mathbb{I}_{\rho_\pi})^r, \mathbb{O}_{\check{M},\rho_\pi}).$$

We expect

$$\mathrm{Gr}_\bullet\mathbb{I}_{\rho_\pi} \sim \nabla_{R\Gamma(\mathrm{Hess}_{I,x})} \in \mathsf{End}(\mathsf{Sym}^\bullet\, R\Gamma(T_x^*\check{M}^{/\!/})),$$

where the latter denotes the derivation induced by
$R\Gamma(\mathrm{Hess}_{I,x}) : R\Gamma(T_x^*\check{M}^{/\!/}) \to R\Gamma(T_x^*\check{M}^{/\!/})$.

Thank you!